

Recent results in four-dimensional non-perturbative string theory ¹

Frank Saueressig ^{1,2}

¹ Service du Physique Théorique, Orme des Merisiers, CEA/Saclay,
91191 Gif-sur-Yvette Cedex, France

² Institute for Theoretical Physics and Spinoza Institute, Utrecht University,
3508 TD Utrecht, The Netherlands

E-mail: Frank.Saueressig@cea.fr

Abstract. We review recent progress in understanding non-perturbative instanton corrections to the hypermultiplet moduli space in type II string compactifications on Calabi-Yau threefolds.

1. Introduction

In recent years it has become clear that non-perturbative effects are an important ingredient for building semi-realistic models connecting string theory to four-dimensional low energy physics. A setup where one might hope to understand such effects in detail is type II string theory compactified on a Calabi-Yau threefold (CY). In this case the low energy physics is captured by a $\mathcal{N} = 2$, $d = 4$ supergravity action and it is conceivable that the constraints coming from string dualities and supersymmetry suffice to determine the perturbative and non-perturbative corrections to the low energy effective action (LEEA), cf. Figure 1.

The classical LEEA can be obtained by carrying out a Kaluza-Klein reduction of ten-dimensional type II supergravity on a CY. The resulting massless modes organize themselves into a supergravity multiplet, n_V vector multiplets (VM) and n_H hypermultiplets (HM). $\mathcal{N} = 2$ supersymmetry implies that the total moduli space \mathcal{M} of these theories factorizes into a local product $\mathcal{M} = \mathcal{M}_{\text{VM}} \otimes \mathcal{M}_{\text{HM}}$ where \mathcal{M}_{VM} and \mathcal{M}_{HM} are parameterized by the scalars of the VM and HM, respectively. Furthermore, \mathcal{M}_{VM} must be special Kähler, i.e., there is a holomorphic prepotential F from which the Kähler potential \mathcal{K} on \mathcal{M}_{VM} can be derived. The hypermultiplet sector \mathcal{M}_{HM} has to be quaternion-Kähler which generically implies that the metric on \mathcal{M}_{HM} does not admit a simple description in terms of a Kähler potential.

Beyond the classical picture the LEEA generically receives quantum corrections from the world sheet conformal field theory (α' -corrections) and in the string coupling constant (g_s -corrections). The α' -corrections encompass a perturbative term and world-sheet instanton corrections from (Euclidean) fundamental strings wrapping holomorphic two-cycles of the CY.

¹ Based on talks given at the European Physical Society HEP 2007 Conference, 19–25 July 2007, Manchester, England and at the Fifth Simons Workshop in Mathematics and Physics, 30 July – 31 August 2007, Stony Brook, USA.

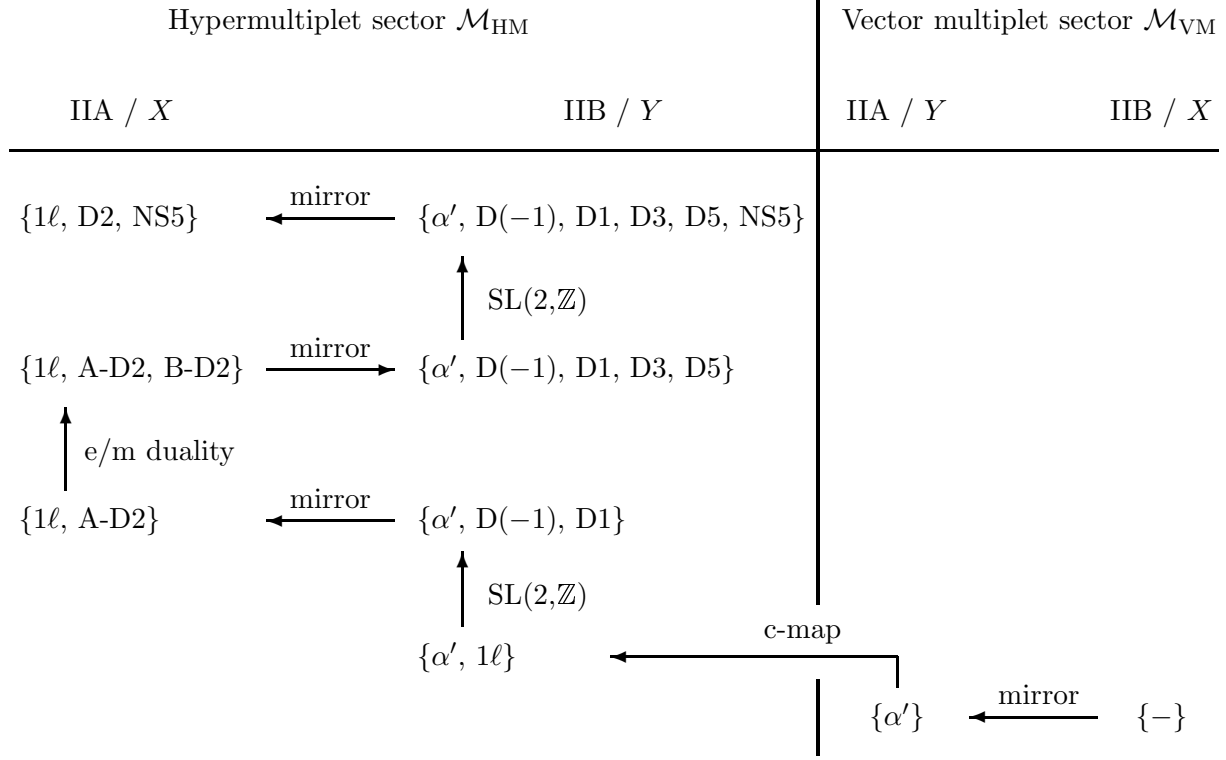


Figure 1. Prospective duality chain for determining the quantum corrected LEEA of type II strings compactified on a generic CY X and its mirror partner Y . In the vector multiplet sector there are α' corrections that appear on the IIA side only and can be obtained via mirror symmetry; they comprise worldsheet loop and instanton corrections. The c-map transfers these into the IIB hypermultiplet sector. In addition, there is a one-loop g_s correction, in the figure denoted by 1ℓ , determined in [1]. Imposing $\text{SL}(2, \mathbb{Z})$ invariance produces the non-perturbative corrections arising from D1-brane and more general (p, q) -string instantons as well as $\text{D}(-1)$ -instantons [3]. The latter naturally combine with the perturbative α' and g_s corrections. As shown in [4], applying mirror symmetry to these corrections gives rise to the A -cycle D2-brane instanton contributions on the IIA side. One might now continue to employ various dualities that should in principle produce the remaining quantum corrections: using electromagnetic (e/m) duality to impose symplectic invariance will give the B -cycle D2-brane instantons. Mirror symmetry will map these to the as of yet unknown D3- and D5-brane instanton corrections on the IIB side. Another application of $\text{SL}(2, \mathbb{Z})$ duality then will give rise to NS5-brane and D5–NS5 bound state instantons. Finally, applying mirror symmetry one last time will produce the NS5-brane corrections on the IIA side. (From [4].)

The perturbative g_s corrections arise from higher genus world sheets and have recently been understood in [1]. Moreover, there are also non-perturbative g_s -corrections coming from Euclidean branes wrapping supersymmetric cycles of the CY [2]. In case of type IIA (even branes) these arise from Euclidean D2-branes (membranes) wrapping special Lagrangian three-cycles of the CY or NS5-branes wrapping the CY itself. For type IIB (odd branes) one has $\text{D}(-1)$ -, D1-, and D3-branes wrapping points, two-, and four-cycles, respectively, together with D5- and NS5-branes wrapping the CY. With respect to these quantum corrections, the factorization of \mathcal{M} has the profound consequence that only those subsectors which contain the dilaton (volume modulus) receive g_s (α') corrections. For both IIA and IIB compactifications the dilaton sits in a hypermultiplet. Thus determining the g_s corrections to the LEEA requires understanding \mathcal{M}_{HM} , on which we will focus in the following.

2. Describing the hypermultiplet sector by tensor multiplets

When studying g_s corrections, it is useful to observe that the spaces \mathcal{M}_{HM} arising from type II string compactifications (at string tree-level) are special in the sense that they admit a $2n_H + 1$ -dimensional Heisenberg algebra of isometries. These shift symmetries reflect the gauge invariance of the ten-dimensional p-forms and remain unbroken by perturbative g_s -corrections. This implies in particular that there are $n_H + 1$ commuting shift symmetries.¹ These isometries can be made manifest by dualizing the corresponding hypermultiplet scalars into tensor fields, so that one deals with $3n_H - 1$ real scalars and $n_H + 1$ rank two antisymmetric tensors.

In type IIB compactifications ($n_H = h^{(1,1)}(Y) + 1$ with $h^{(p,q)}(Y)$ denoting the Hodge numbers of Y), the $h^{(1,1)}(Y) + 2$ tensor fields arise naturally as the space-time parts of the NS and RR two-form and the expansion of the RR four-form with respect to the $h^{(1,1)}$ harmonic two-forms of Y . The tensor multiplet spectrum is completed by $3h^{(1,1)}(Y) + 2$ scalars comprising the ten-dimensional axion-dilaton $\tau = \tau_1 + i\tau_2 = C_0 + ie^{-\phi}$ and $3h^{(1,1)}$ scalars coming from the period integrals of the Kähler form J and the NS and RR 2-forms with respect to a basis of two-cycles $\gamma_{(2)}^a \in H^2(Y, \mathbb{Z})$

$$z^a = b^a + it^a = \int_{\gamma_{(2)}^a} B_{\text{NS}} + iJ, \quad c^a = \int_{\gamma_{(2)}^a} C_2, \quad a = 2, \dots, h^{(1,1)}(Y) + 1, \quad (1)$$

Utilizing the tensor multiplet picture, \mathcal{M}_{HM} can elegantly be described by combining projective superspace and superconformal calculus techniques. The underlying idea is that \mathcal{M}_{HM} with $n_H + 1$ commuting isometries has an equivalent formulation in terms of rigidly superconformal tensor multiplet (TM) Lagrangians [5]

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \oint_{\mathcal{C}} \frac{d\zeta}{2\pi i \zeta} H(\eta^I), \quad I = 0, \dots, h^{(1,1)}(Y) + 1. \quad (2)$$

Here $\eta^I = v^I/\zeta + x^I - \zeta\bar{v}^I$ denotes a $\mathcal{N} = 2$ tensor superfield, $H(\eta^I)$ is a weakly homogeneous function of degree one (i.e., logarithmic terms are allowed) with no explicit ζ -dependence, \mathcal{C} is an arbitrary contour in the complex ζ -plane, and $\int d^2\theta d^2\bar{\theta}$ is the superspace measure. Coupling the Lagrangian (2) to conformal supergravity (Weyl multiplet) and carrying out the superconformal quotient determines \mathcal{M}_{HM} [6, 7].

When taken as a function of the scalar fields, $\mathcal{L}(v^I, \bar{v}^I, x^I)$ gives rise to the tensor potential [7]

$$\chi = -\mathcal{L} + x^I \frac{\partial \mathcal{L}}{\partial x^I}, \quad (3)$$

which also fixes \mathcal{M}_{HM} . In fact carrying out the Legendre transform by setting $w_I + \bar{w}_I = \frac{\partial \mathcal{L}}{\partial x^I}$ and taking $x^I = x^I(v, \bar{v}, w + \bar{w})$, χ becomes the Kähler potential on the hyperkähler cone (or Swann bundle) over \mathcal{M}_{HM} [6]. Describing \mathcal{M}_{HM} through the functions \mathcal{L}, χ is in close analogy to the vector multiplet sector whose couplings are completely encoded in F, \mathcal{K} .

In the superconformal formulation, the physical fields (1) are given by (SU(2) and dilatation invariant) functions of the tensor multiplet scalars [8]

$$\tau = \frac{1}{(\vec{r}^0)^2} (\vec{r}^0 \cdot \vec{r}^1 + i |\vec{r}^0 \times \vec{r}^1|), \quad z^a = \frac{\eta_+^a}{\eta_+^1}, \quad c^a = \frac{(\vec{r}^0 \times \vec{r}^1) \cdot (\vec{r}^1 \times \vec{r}^a)}{|\vec{r}^0 \times \vec{r}^1|^2}, \quad (4)$$

¹ These $n_H + 1$ shift symmetries are still preserved once the non-perturbative corrections from D(-1)- and D1-instantons in type IIB or their mirror configurations, A-type membrane instantons in type IIA are included.

where $\vec{r}^I = [2v^I, 2\bar{v}^I, x^I]$ with $\vec{r}^I \cdot \vec{r}^J = 2v^I \bar{v}^J + 2v^J \bar{v}^I + x^I x^J$ and $\eta_+^\Lambda = \eta^\Lambda(\zeta_+)$, $\Lambda = \{1, a\}$, where $\zeta_+ = \frac{1}{2\bar{v}^0}(x^0 - |\vec{r}^0|)$ is one of the roots of $\zeta\eta^0(\zeta)$. Working with the tensor potential has the virtue that (up to a conformal factor $r^0 = |\vec{r}^0|$) χ can be completely expressed in terms of the physical fields (4), which considerably simplifies the physical interpretation of the quantum corrections. Moreover, the close relation between χ and the Kähler potential on the hyperkähler cone over \mathcal{M}_{HM} implies that (discrete) symmetries of \mathcal{M}_{HM} as, e.g., the $\text{SL}(2, \mathbb{Z})$ invariance of the type IIB string, are reflected by the invariance of χ .

Let us close this section by remarking that superconformal invariance of the TM Lagrangian implies that χ is subject to a set of partial differential equations. These are automatically solved if χ is obtained via the contour formulation (2), but can also be used to verify supersymmetry without resorting to \mathcal{L} . For one HM one finds a single constraint on χ which is equivalent to the condition for four-dimensional quaternion-Kähler spaces with two commuting isometries found by Calderbank and Pedersen [9]. The three partial differential equations and metric arising in the two HM case have been worked out in [10]. These supersymmetry constraints will probably be an important ingredient in demonstrating the uniqueness of the modular completion below.

3. Instanton corrections to the hypermultiplet moduli space

In order to determine the g_s -corrections to \mathcal{M}_{HM} , we follow the strategy outlined in Figure 1. Building on the off-shell description of the c-map [11] the perturbatively corrected hypermultiplet moduli space has been determined in [1]

$$\mathcal{L}(v, \bar{v}, x) = \text{Im} \oint_{\mathcal{C}_0} \frac{d\zeta}{2\pi i \zeta} \frac{F(\eta^\Lambda)}{\eta^0} \pm \frac{i 2\zeta(2) \chi_E}{(2\pi)^3} \oint_{\mathcal{C}_1} \frac{d\zeta}{2\pi i \zeta} \eta^0 \ln(\eta^0). \quad (5)$$

Here $F(\cdot)$ is the holomorphic prepotential of the T-dual type IIA vector multiplet geometry, η^0 is an additional TM acting as a conformal compensator and the contours \mathcal{C}_0 and \mathcal{C}_1 enclose the root ζ_+ of $\zeta\eta^0$ and the branch cut between $\zeta = 0$ and ζ_+ respectively [12]. The first term encodes the classical c-map including the tree-level α' corrections to \mathcal{M}_{HM} . The second term is the universal one-loop correction with the upper (lower) sign corresponding to type IIB (type IIA) strings and χ_E being the Euler number of the CY.

Substituting (5) into (3) yields the tensor potential underlying the perturbatively corrected \mathcal{M}_{HM} . For type IIB the corresponding expression naturally splits into a classical part χ_{cl} , a piece containing the perturbative α' and g_s corrections χ_{pt} , and χ_{ws} capturing the world-sheet instanton contributions

$$\begin{aligned} \chi_{\text{cl}} &= 4 r^0 \tau_2^2 \frac{1}{3!} \kappa_{abc} t^a t^b t^c, & \chi_{\text{pt}} &= \frac{1}{(2\pi)^3} r^0 \chi_E [\zeta(3) \tau_2^2 + 2\zeta(2)] , \\ \chi_{\text{ws}} &= - \frac{r^0 \tau_2^2}{(2\pi)^3} \sum_{k_a} n_{k_a} \left[\text{Li}_3(e^{2\pi i k_a z^a}) + 2\pi k_a t^a \text{Li}_2(e^{2\pi i k_a z^a}) + c.c. \right] . \end{aligned} \quad (6)$$

Here κ_{abc} are triple intersection numbers and n_{k_a} the Gopakumar-Vafa invariants of Y . The one-loop correction is suppressed by $\tau_2^{-2} = g_s^2$ compared to the three-level terms and gives rise to the second term in χ_{pt} .

In order to determine the D(-1)- and D1-brane instanton corrections we make use of the non-perturbative $\text{SL}(2, \mathbb{Z})$ invariance of the type IIB string. The modular transformation properties of the four-dimensional scalars are inherited from the ten-dimensional fields through the dimensional reduction [13]

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad t^a \mapsto t^a |c\tau + d|, \quad b^a \mapsto db^a + c c^a, \quad c^a \mapsto b b^a + a c^a. \quad (7)$$

IIB HM	SL(2, \mathbb{Z})-invariant	IIA HM	composed from IIB terms
χ_{cl}	$= \chi_{\text{cl}}$	χ_{tree}	$= \chi_{\text{cl}} + \chi_{\text{ws-pert}} + \chi_{\text{ws-inst}}$
$\chi_{(-1)}$	$= \chi_{\text{ws-pert}} + \chi_{\text{loop}} + \chi_{D(-1)}$	χ_{loop}	$= \chi_{\text{loop}}$
$\chi_{(1)}$	$= \chi_{\text{ws-inst}} + \chi_{D1}$	χ_{A-D2}	$= \chi_{D(-1)} + \chi_{D1}$

Table 1. Rearrangement of the type IIB tensor potential under mirror symmetry.

The modular transformation of the conformal compensator $r^0 \mapsto r^0 |c\tau + d|$ is determined by lifting the action of $\text{SL}(2, \mathbb{Z})$ to superspace [14].

Applying the transformations (7) to (6) one finds that χ_{cl} is modular invariant while the α' and g_s corrections break the $\text{SL}(2, \mathbb{Z})$ symmetry. Restoring $\text{SL}(2, \mathbb{Z})$ invariance by a modular completion of χ_{pt} and χ_{ws} gives [3]

$$\begin{aligned}\chi_{(-1)}^{\text{IIB}} &= \frac{r^0 \tau_2^{1/2}}{2(2\pi)^3} \chi_E \sum'_{m,n} \frac{\tau_2^{3/2}}{|m\tau + n|^3} , \\ \chi_{(1)}^{\text{IIB}} &= -\frac{r^0 \tau_2^{1/2}}{(2\pi)^3} \sum_{k_a} n_{k_a} \sum'_{m,n} \frac{\tau_2^{3/2}}{|m\tau + n|^3} (1 + 2\pi |m\tau + n| k_a t^a) e^{-S_{m,n}} .\end{aligned}\tag{8}$$

Besides the α' and perturbative g_s corrections this result also includes the instanton corrections from D(-1)- and D1-branes, respectively. In particular one notes the appearance of the (m, n) -string instanton action

$$S_{m,n} = 2\pi k_a (|m\tau + n| t^a - im c^a - in b^a) ,\tag{9}$$

where m and n are the units of D1 and fundamental string charge.

By the SYZ construction of mirror symmetry [15], the D-brane instanton corrections in (8) are mirror to Euclidean D2-branes, wrapping so-called A-cycles of the mirror CY. The corresponding mirror map in the hypermultiplet sector is given by [13, 4]

$$\phi_{\text{IIA}} = \phi_{\text{IIB}} , \quad A^1 = \tau_1 , \quad A^a = -(c^a - \tau_1 b^a) , \quad z_{\text{IIA}}^a = z_{\text{IIB}}^a .\tag{10}$$

Here the A^Λ arise as the periods of the ten-dimensional RR three-form over (electric) A-cycles, while z_{IIA}^a are the complex structure moduli of the mirror manifold X .² Implementing the mirror map requires extracting the contributions from D(-1)- and D1-branes from (8) following Table 1. The resulting correction term due to D2-brane instantons was then given in [4]

$$\chi_{A-D2}^{\text{IIA}} = -\frac{r^0 \tau_2}{2\pi^2} \sum_{k_\Lambda} n_{k_\Lambda} \sum_{m \neq 0} \frac{|k_\Lambda z^\Lambda|}{|m|} K_1(2\pi \tau_2 |m k_\Lambda z^\Lambda|) e^{-2\pi i m k_\Lambda A^\Lambda} .\tag{11}$$

Here, $k_\Lambda = (n, k_a)$, $z^\Lambda = (1, z^a)$, $A^\Lambda = (A^1, A^a)$, and the sum over k_a now includes the zero-vector $k_a = 0$, but $k_\Lambda = 0$ is excluded. The type IIA instanton numbers are

$$n_{(n, k_a=0)} = \frac{1}{2} \chi_E(X) , \quad n_{(n, k_a)} = n_{k_a}(Y) \quad \text{as in type IIB} .\tag{12}$$

Note that here $n_{k_a}(Y)$ are the Gopakumar-Vafa invariants of the *mirror* CY. This indicates a deep connection between the properties of holomorphic two-cycles of Y (counted by $n_{k_a}(Y)$) and special Lagrangian three-cycles of the mirror manifold X .

² Note that these relations require a special choice of symplectic basis in $H^3(X, \mathbb{Z})$, which induces a distinction between A- and B-cycles based on their properties under mirror symmetry, see [15] for details.

One particular limit of eq. (11) is the so-called conifold limit considered by Ooguri and Vafa [16]. In this limit one particular complex structure modulus and the string coupling constant are sent to zero, $z^* \rightarrow 0$, $g_s = \tau_2^{-1} \rightarrow 0$, while keeping the product $\tau_2 z^* = \lambda$ and all other complex structure moduli z^Λ , $\Lambda \neq *$ finite. In this limit gravity decouples and (11) gives rise to a four-dimensional hyperkähler metric. It was then shown in [17] that the resulting hyperkähler metric is precisely the one found in [16] by evoking symmetry and regularity arguments.

The instanton corrections discussed above, however, do not determine the full quantum corrected LEEA, as the corrections from D2-brane instantons wrapping B-cycles in type IIA (or their mirror configurations from the D3- and D5-brane instantons in type IIB) and the contributions from NS5-branes have not been included. These instanton corrections are beyond the scope of the tensor multiplet formalism of Section 2, as they will break (some of) the isometries made manifest by the tensors. Nevertheless, the superconformal formulation based on the Kähler potential on a generic hyperkähler cone [6] is still applicable and expected to play a key role in completing the implementation of the duality chain in Figure 1.

Acknowledgements

It's a pleasure to thank B. de Wit, D. Robles-Llana, M. Roček, U. Theis and S. Vandoren for fruitful collaboration. Furthermore, I would like to thank the organizers of the European Physical Society HEP 2007 conference and the Fifth Simons Workshop in Mathematics and Physics for their hospitality and S. Alexandrov, T. Grimm, S. Sethi, A. Klemm and C. Vafa for interesting discussions. This work is partially supported by the European Commission Marie Curie Fellowship no. MEIF-CT-2005-023966 and the grant “Structure of vacuum, topological strings and black holes” from the French *Agence Nationale de la Recherche*.

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